Gauge Formulation for Higher Order Gravity

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Abstract

This work is an application of the second order gauge theory for the Lorentz group, where a description of the gravitational interaction is obtained which includes derivatives of the curvature. We analyze the form of the second field strenght, $G = \partial F + fAF$, in terms of geometrical variables. All possible independent Lagrangians constructed with quadratic contractions of F and quadratic contractions of G are analyzed. The equations of motion for a particular Lagrangian, which is analogous to Podolsky's term of his Generalized Electrodynamics, are calculated.

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1 Introduction

At present there are many proposals to modify gravitation in order to solve several problems as the present day accelerated expansion of the universe [1,2], or to accommodate corrections of quantum nature which arise from the classical effective backreaction of quantum matter in a curved background [3]. Effective action is widely used in quantum field theory as a powerful method of calculation. The Podolsky generalized electrodynamics, for instance, can be viewed as an effective description of quantum correction to the classical Maxwell Lagrangian [4].

For gravitation, usually higher orders terms are introduced by means of Lagrangian contributions quadratic in the Riemann tensor and their contractions [5]. This is inspired by 1-loop corrections in the Einstein-Hilbert action in the quantized weak field approximation, or in the equivalent Feynman construction of a spin-2 field on the flat Minkowski background [6]. Besides this, at the quantum level, the S matrix for the Einstein theory is finite at one-loop level, but diverges at the two-loop order [7], which motivate the introduction of derivative terms in the Riemann tensor for the action [8].

On the other hand, recently was proposed a second order construction of gauge theories based on Utiyama's approach [9], which gives exactly the same correction terms as in the Podolsky electrodynamics, but now arising from the principle of local gauge invariance [10]. Therefore, a connection between quantum corrections and gauge higher order terms in the action was conjectured, which was proved be fulfilled also for the effective Alekseev-Arbuzov-Baikov lagrangian of the infrared regime of QCD [11].

Here, we analyse the gauge formulation of the gravitational field based on the framework of the second order gauge theory. The simplest gauge group is given by the Lorentz homogeneous group in the context of a Riemannian description of the gravitational field. Since the gauge field is given in such case by the local spin connection, higher order in the gauge field involves naturally the derivative of the curvature tensor. In this sense, the actual higher order gravitational lagrangian should be constructed from invariants using the covariant derivative of the Riemann tensor instead of the usual quadratic terms in curvature.

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The relationship between the algebraic gauge description and the geometrical one is settled by means of the introduction of the tetrad field, and the construction of the covariant derivatives associated with the both symmetries: the local Lorentz and the global diffeomorphic coordinate transformations.

The paper is structured as follows. In section 2 we review some results relating gauge invariance and gravitation discussing the geometrization of this interaction and the rising of covariant derivatives operator (under gauge transformations and general coordinate transformations). We use latin indexes, $a, b, ...$, for the internal Lorentz group and greek indexes for the tangent space of the space-time manifold.

The field strengths F and G of the second order treatment are introduced in section 3, where they are also written in their geometrical counterparts: the Riemann curvature tensor and its covariant derivative.

Section 4 deals with the possible quadratic invariants of the type F^2 and G^2. All the possible contractions are studied and only the independent invariants are kept. In the following section, section 5, these invariants are shown to satisfy the identity which restricts the theories that may be called of the gauge type.

Among all invariants, we select $L_P = \frac{1}{2}h\ \delta^\rho R_{\rho\chi}\delta_\mu R^{\mu\chi}$, the Podolsky-like Lagrangian, for calculating the equation of motion of the gravitational field. This higher order gravity application is done in section 6.

Final remarks are given in section 7.

2 Gauge Interaction and Covariance

In 1956 Utiyama [9] has shown how to implement a gauge description for gravitational interaction with matter fields $Q^A(x)$ transforming according to

$$\delta Q^A(x) = \frac{1}{2}\varepsilon^{ab}(x)\left(\Sigma_{ab}\right)^A_B Q^B \ , \tag{1}$$

as a implementation of the local invariance exigency of the action under continuous proper Lorentz transformations, which are characterized by the generators Σ_{ab} satisfying the operation of a typical Lie group,

$$[\Sigma_{ab}, \Sigma_{cd}] = \frac{1}{2}f_{ab,\ cd}^{\ ef}\Sigma_{ef} \ , \tag{2}$$

where

$$f_{ab,\ cd}^{\ ef} = \left\{\left[\eta_{bc}\delta^e_a - \eta_{ac}\delta^e_b\right]\delta^f_d - \left[\eta_{bd}\delta^e_a - \eta_{ad}\delta^e_b\right]\delta^f_c\right\} - e \leftrightarrow f$$

are the structure constants obeying the Jacobi identity. $\varepsilon^{ab} = -\varepsilon^{ba}$ are the parameters of the local transformation. The capital latin indexes are for the components of the matter field.

It was clearly shown that it is necessary to introduce the compensating field $\omega_\mu^{ab}(x)$ transforming as a connection,

$$\delta\omega^{ef}_{\ \ \mu} = \frac{1}{4}\varepsilon^{ab}(x) f_{ab,\ cd}^{\ \ ef}\,\omega^{cd}_{\ \ \mu} + \partial_\mu \varepsilon^{ef}(x)\ . \tag{3}$$

The invariance of the theory implies that the compensating field must appear through the *gauge covariant derivative*

$$D_\mu Q^A \equiv \partial_\mu Q^A - \frac{1}{2}\omega^{ab}_{\ \ \mu}\left(\Sigma_{ab}\right)^A_{\ B} Q^B \tag{4}$$

i.e.,

$$\delta D_\mu Q^A = \frac{1}{2}\varepsilon^{ab}\left(\Sigma_{ab}\right)^A_{\ B} D_\mu Q^B\ . \tag{5}$$

The equations of motion for the Q field are then given by:

$$\frac{\partial\mathcal{L}}{\partial Q^A} - D_\mu \frac{\partial\mathcal{L}}{\partial\left(D_\mu Q^A\right)} \equiv 0\ .$$

2.1 Geometrizing the description

For instance, consider as the matter field a Lorentz tensor of rank 2, Q^{ij}. In this case [12]

$$\left(\Sigma_{ab}\right)^{ij}_{\ kl} = \eta_{bk}\delta^i_{\ a}\delta^j_{\ l} - \delta^i_{\ b}\eta_{ak}\delta^j_{\ l} + \delta^j_{\ a}\eta_{bl}\delta^i_{\ k} - \delta^j_{\ b}\eta_{al}\delta^i_{\ k}$$

and the gauge covariant derivative reads

$$D_\mu Q^{ij} = \partial_\mu Q^{ij} - \omega^{ib}_{\ \ \mu}\eta_{bk}Q^{kj} - \omega^{jb}_{\ \ \mu}\eta_{bk}Q^{ik}\ .$$

Now, we introduce the geometrizing tetrad fields, $h^j_{\ \nu}$, which transform Lorentz indices in space-time ones,

$$Q^{i\mu} \equiv h_j^{\ \mu} Q^{ij},$$

and vice-versa,

$$Q^{ij} = h^j_{\ \nu} Q^{i\nu}\ .$$

The components $h^j_{\ \nu}$ are the inverse of $h_j^{\ \mu}$,

$$h^j_{\ \nu} h_j^{\ \mu} = \delta^\mu_{\ \nu},\ \ h^i_{\ \nu} h_j^{\ \nu} = \delta^i_{\ j}\ ,$$

in such way that the space-time metric, $g_{\mu\nu}$, is related with η_{ij}:

$$g_{\mu\nu} = \eta_{ij} h^i_{\ \nu} h^j_{\ \mu} \ , \quad \eta_{ij} = h_i^{\ \mu} h_j^{\ \nu} g_{\mu\nu} \ ,$$
$$h = \sqrt{\det h^j_\mu} = \sqrt{-g} \ .$$

We define

$$\nabla_\mu Q^{i\nu} \equiv h_j^\nu D_\mu Q^{ij} = D_\mu Q^{i\nu} + h_j^\nu D_\mu h^j_{\ \lambda} Q^{i\lambda} \ , \tag{6}$$

which can be rewritten as a *total covariant derivative*,

$$\nabla_\mu Q^{i\nu} = \partial_\mu Q^{i\nu} - \omega^{ib}_{\ \ \mu} \eta_{bk} Q^{k\nu} + \tilde{\Gamma}^\nu_{\ \mu\alpha} Q^{i\alpha} \ , \tag{7}$$

if one defines

$$\tilde{\Gamma}^\nu_{\ \mu\alpha} \equiv h_j^\nu \left(D_\mu h^j_{\ \alpha} \right) \ . \tag{8}$$

This definition can be rewritten as

$$\nabla_\mu h^j_{\ \alpha} \equiv 0 \ , \tag{9}$$

which is the geometrical statement of absolute parallelism of the tetrad field.

One can also find a relation between covariant derivative defined on the space-time manifold and the gauge covariant derivative defined on the fiber (or, equivalently, between space-time covariant derivative and the total covariant one) as an extension of the relation (6):

$$Q^{\lambda\mu} \equiv h_i^\lambda h_j^\mu Q^{ij} \ , \quad Q^{ij} = h^i_{\ \lambda} h^j_{\ \nu} Q^{\lambda\nu} \ ; \tag{10}$$

$$\begin{aligned} \delta_\mu Q^{\lambda\nu} &\equiv h_i^\lambda h_j^\nu D_\mu Q^{ij} = h_i^\lambda \nabla_\mu Q^{i\nu} = \\ &= \partial_\mu Q^{\lambda\nu} + \tilde{\Gamma}^\lambda_{\ \mu\beta} Q^{\beta\nu} + \tilde{\Gamma}^\nu_{\ \mu\alpha} Q^{\lambda\alpha} \ . \end{aligned} \tag{11}$$

On the other hand, (6) can be formally rearranged in

$$\delta_\mu Q^{\lambda\nu} = \nabla_\mu \left(h_i^\lambda Q^{i\nu} \right)$$

by virtue of the absolute parallelism of the tetrad field.

The relation between the connection (8) and the space-time one is established linking the above equations to the metricity condition. First, we notice the following identity

$$D_\mu \eta^{ij} = -\omega^{ij}_{\ \ \mu} - \omega^{ji}_{\ \ \mu} = 0 \ ,$$

where we used Cartesian coordinates with $\eta = diag\,(1, -1, -1, -1)$. From (11), it results

$$\delta_\mu g^{\alpha\beta} = 0 \ ,$$

the metricity condition. Once the space-time Γ is *defined* as a *metric compatible* connection, we are left with

$$\Gamma^{\nu}_{\mu\alpha} = \tilde{\Gamma}^{\nu}_{\mu\alpha} \ , \tag{12}$$

the equality of the connection (8) and the space-time Γ. This equivalence shows that the gauge covariant derivative of the vierbein field can *not* be null. If $D_\mu h^j_\alpha$ was zero, the space-time connection would also be null according with (8) and this would restrict the theory to the case of a flat manifold.

Another proof of the identity (12) is obtained as follows. Consider the space-time derivative of the metric tensor,

$$\delta_\mu g^{\lambda\nu} \equiv h_i^{\lambda} h_j^{\nu} D_\mu \eta^{ij} \ .$$

By the definition of the total covariant derivative, this is the same as

$$\delta_\mu g^{\lambda\nu} = h_i^{\lambda} \nabla_\mu \left(h_j^{\nu} \eta^{ij} \right) = h_i^{\lambda} \nabla_\mu h^{i\nu} \equiv 0 \ ,$$

in view of the tetrad absolute parallelism (9). Therefore, the derivative $\delta_\mu \left[\tilde{\Gamma}\right]$ is metric compatible, which leads to (12). This second argument has the advantage of being independent of the choice of the local system of reference on the fiber, e.g., the coordinates may not be Cartesian.

We will restrict our analysis to a symmetric space-time connection in order to approach the riemannian description. The extension to the Riemann-Cartan case is quite natural, but would imply different types of invariants as admissible Lagrangians (see discussion bellow).

2.1.1 Covariance of the Total Derivative

From [9] we know that D is covariant under gauge transformations, as dictated by (5). Now, we will proof that the total covariant derivative is covariant under gauge as well as under coordinate transformations. Let us begin analysing Lorentz transformation for a rank two Lorentz tensor,

$$\begin{aligned} \delta\nabla_\mu Q^{i\nu} &= \delta h_j^{\nu} D_\mu Q^{ij} + h_j^{\nu} \delta D_\mu Q^{ij} = \\ &= \frac{1}{2}\varepsilon^{ab} \left(\Sigma_{ab}\right)^k_{\ j} h_k^{\nu} D_\mu Q^{ij} + \varepsilon^{ab} \left(\delta^i_{\ a} \eta_{bk} h_j^{\nu} D_\mu Q^{kj} + \delta^j_{\ a} \eta_{bl} h_j^{\nu} D_\mu Q^{il} \right) \ . \end{aligned}$$

Using the vectorial spin representation

$$\left(\Sigma_{ab}\right)^k_{\ j} = \eta_{aj} \delta^k_{\ b} - \eta_{bj} \delta^k_{\ a}$$

we find

$$\delta\nabla_\mu Q^{i\nu} = \frac{1}{2}\varepsilon^{ab} \left(\Sigma_{ab}\right)^i_{\ k} h_j^{\nu} D_\mu Q^{kj} \ ,$$

which proves the gauge covariance in this case. For a tensor of other rank,or spinorial fields, one proceeds in a similar way.

Under general coordinate transformations,

$$\bar{x} = x + \delta x \ ,$$

each component of the matter field is invariant:

$$\bar{\delta} Q^{A} \equiv \bar{Q}^{A}\left(\bar{x}\right) - Q^{A}\left(x\right) = 0 \ .$$

The infinitesimal changes in space-time vectors and covectors are given by

$$\begin{aligned} \bar{\delta} V^{\mu}\left(x\right) &= \bar{V}^{\mu}\left(\bar{x}\right) - V^{\mu}\left(x\right) = \frac{\partial \delta x^{\mu}}{\partial x^{\nu}} V^{\nu} \ , \\ \bar{\delta} V_{\mu}\left(x\right) &= \bar{V}_{\mu}\left(\bar{x}\right) - V_{\mu}\left(x\right) = -\frac{\partial \delta x^{\nu}}{\partial \bar{x}^{\mu}} V_{\nu} \ . \end{aligned}$$

We will use these on (6). We have

$$\bar{\delta} \nabla_{\mu} Q^{i\nu} = \bar{\delta} h_{j}^{\nu} D_{\mu} Q^{ij} + h_{j}^{\nu} \bar{\delta} D_{\mu} Q^{ij} \ .$$

But as

$$\bar{\delta} h_{j}^{\nu} = \frac{\partial \delta x^{\nu}}{\partial x^{\mu}} h_{j}{}^{\mu}; \qquad \bar{\delta} D_{\mu} Q^{ij} = -\frac{\partial \delta x^{\lambda}}{\partial x^{\mu}} D_{\lambda} Q^{ij} \ ,$$

it results

$$\bar{\delta} \nabla_{\mu} Q^{i\nu} = \frac{\partial \delta x^{\nu}}{\partial x^{\lambda}} \nabla_{\mu} Q^{i\lambda} - \frac{\partial \delta x^{\lambda}}{\partial x^{\mu}} \nabla_{\lambda} Q^{i\nu} \ .$$

On the other hand, the transformation rule for a mixed tensor of rank two is:

$$\bar{\delta} A_{\mu}^{\nu} = \frac{\partial \delta x^{\nu}}{\partial x^{\lambda}} A_{\mu}^{\lambda} - \frac{\partial \delta x^{\alpha}}{\partial x^{\mu}} A_{\alpha}^{\nu} \ ,$$

from which follows the ordinary infinitesimal transformation for the tensor $\nabla_{\mu} Q^{i\nu}$.

Above we showed the covariance relations for tensors of particular ranks. However, the result is valid for objects of higher ranks since they can be construct from tensorial product of tensors of lower orders. The spinorial case is even simpler: D_{μ} is the only derivative operator that makes any sense on acting upon this field.

3 Gauge Field Lagrangian

The basic hypothesis we will assume is: the Lagrangian for the free gauge potential depends on the field, its first and second order derivatives and $L_0 = L_0\left(\omega^{ef}_{\ \ \mu}, \partial_\nu \omega^{ef}_{\ \ \mu}, \partial_\rho \partial_\nu \omega^{ef}_{\ \ \mu}\right)$ obeys local invariance principle under (3). This enable us to use the results presented elsewhere [10] to construct a gauge formulation for higher order gravitation theories.

3.1 The field strengths

According to the work [10], we can reexpress

$$\delta L_0 = \frac{1}{2}\frac{\partial L_0}{\partial \omega^{ef}_{\ \ \mu}}\delta\omega^{ef}_{\ \ \mu} + \frac{1}{2}\frac{\partial L_0}{\partial\left(\partial_\nu \omega^{ef}_{\ \ \mu}\right)}\delta\partial_\nu\omega^{ef}_{\ \ \mu} + \frac{1}{2}\frac{\partial L_0}{\partial\left(\partial_\rho\partial_\nu \omega^{ef}_{\ \ \mu}\right)}\delta\partial_\rho\partial_\nu\omega^{ef}_{\ \ \mu} \equiv 0 \ ,$$

splitting it into a set of four hierarchical equations after substituting (3) and claiming the independence of the parameters ε^{ab} and their derivatives. Three of these functional equations are used to conclude that

$$L_0 = L_0\left(F, G\right); \qquad \frac{\partial L_0}{\partial \omega^{ab}_{\ \ \mu}} \equiv 0 \ , \tag{13}$$

where

$$F^{ab}_{\ \ \mu\nu} = \partial_\mu \omega^{ab}_{\ \ \nu} - \partial_\nu \omega^{ab}_{\ \ \mu} - \eta_{cd}\omega^{ac}_{\ \ \mu}\omega^{db}_{\ \ \nu} + \eta_{cd}\omega^{ac}_{\ \ \nu}\omega^{db}_{\ \ \mu} \tag{14}$$

and

$$G^{ab}_{\ \ \beta\rho\sigma} = D_\beta F^{ab}_{\ \ \rho\sigma} = \partial_\beta F^{ab}_{\ \ \rho\sigma} - \eta_{fd}\omega^{af}_{\ \ \beta}F^{db}_{\ \ \rho\sigma} + \eta_{fd}\omega^{af}_{\ \ \beta}F^{bd}_{\ \ \rho\sigma} \ . \tag{15}$$

The remaining hierarchical equation put in terms of the gauge fields F and G,

$$\frac{\partial L_0}{\partial F^{ad}_{\ \ \rho\sigma}} f_{bc\,gh}^{\ \ ad} F^{gh}_{\ \ \rho\sigma} + \frac{\partial L_0}{\partial G^{ad}_{\ \ \beta\rho\sigma}} f_{bc\ gh}^{\ \ ad} G^{gh}_{\ \ \beta\rho\sigma} \equiv 0 \ , \tag{16}$$

imposes restrictions upon the functional form eventually chosen for L_0. Substituting the structure constants, this condition can be explicitly written as

$$\frac{\partial L_0}{\partial F^{ad}_{\ \ \rho\sigma}}\left[\eta_{cg}\delta^a_{\ b} - \eta_{bg}\delta^a_{\ c}\right] F^{gd}_{\ \ \rho\sigma} + \frac{\partial L_0}{\partial G^{ad}_{\ \ \beta\rho\sigma}}\left[\eta_{cg}\delta^a_{\ b} - \eta_{bg}\delta^a_{\ c}\right] G^{gd}_{\ \ \beta\rho\sigma} \equiv 0 \ . \tag{17}$$

3.2 Geometrical variables

In this section we will show how to interpret all objects and condition of the previous sections in terms of a geometrical point of view. From (9) we read

$$\omega^{eg}_{\sigma} = \eta^{gc} h_c^{\ \alpha} \left(\partial_\sigma h^e_{\ \alpha} - \Gamma^\nu_{\ \sigma\alpha} h^e_{\ \nu}\right)$$

and therefore the field strength F is written as

$$F^{eg}_{\ \ \beta\sigma} = \eta^{gc} h_c^{\ \alpha} h^e_{\ \gamma} \left[\partial_\sigma \Gamma^\gamma_{\ \beta\alpha} - \partial_\beta \Gamma^\gamma_{\ \sigma\alpha} + \Gamma^\nu_{\ \beta\alpha} \Gamma^\gamma_{\ \sigma\nu} - \Gamma^\nu_{\ \sigma\alpha} \Gamma^\gamma_{\ \beta\nu}\right] \ ,$$

where we recognize the expression of the Riemann tensor [13],

$$R_{\sigma\beta\alpha}^{\ \ \ \ \gamma} \equiv \partial_\sigma \Gamma^\gamma_{\ \beta\alpha} - \partial_\beta \Gamma^\gamma_{\ \sigma\alpha} + \Gamma^\nu_{\ \beta\alpha} \Gamma^\gamma_{\ \sigma\nu} - \Gamma^\nu_{\ \sigma\alpha} \Gamma^\gamma_{\ \beta\nu},$$

i.e.,

$$F^{eg}_{\ \ \beta\sigma} = \eta^{gc} h_c^{\ \alpha} h^e_{\ \gamma} R_{\sigma\beta\alpha}^{\ \ \ \ \gamma} \ . \tag{18}$$

The easiest way to finding the geometrical counterpart of G is to apply the geometrizing relations (10,11):

$$h_a^{\ \mu} h_b^{\ \nu} G^{ab}_{\ \ \beta\rho\sigma} = h_a^{\ \mu} h_b^{\ \nu} D_\beta F^{ab}_{\ \ \rho\sigma} = \delta_\beta F^{\mu\nu}_{\ \ \rho\sigma} \ ;$$

$$F^{\mu\nu}_{\ \ \rho\sigma} = h_a^{\ \mu} h_b^{\ \nu} F^{ab}_{\ \ \rho\sigma}$$

and use (18). We arrive at

$$G^{ab}_{\ \ \beta\rho\sigma} = h^a_{\ \mu} h^b_{\ \nu} g^{\nu\alpha} \delta_\beta R_{\sigma\rho\alpha}^{\ \ \ \ \mu} \ , \tag{19}$$

which is the most natural equation one would expect in view of the relation (15) between F and G.

By means of the geometrical descriptions (18) and (19), we are able to find

$$\begin{aligned}
\frac{\partial L_0}{\partial F^{ad}_{\ \ \rho\sigma}} &= \frac{\partial L_0}{\partial R_{\lambda\beta\alpha}^{\ \ \ \ \gamma}} \frac{\partial R_{\lambda\beta\alpha}^{\ \ \ \ \gamma}}{\partial F^{ad}_{\ \ \rho\sigma}} = \frac{\partial L_0}{\partial R_{\sigma\rho\alpha}^{\ \ \ \ \gamma}} \eta_{bd} h^b_{\ \alpha} h_a^{\ \gamma}, \\
\frac{\partial L_0}{\partial G^{ad}_{\ \ \beta\rho\sigma}} &= \frac{\partial L_0^{(4)}}{\partial \left(\delta_\lambda R_{\gamma\nu\alpha}^{\ \ \ \ \mu}\right)} \frac{\partial \left(\delta_\lambda R_{\gamma\nu\alpha}^{\ \ \ \ \mu}\right)}{\partial G^{ad}_{\ \ \beta\rho\sigma}} = \frac{\partial L_0}{\partial \left(\delta_\beta R_{\sigma\rho\alpha}^{\ \ \ \ \mu}\right)} g_{\alpha\omega} h_a^{\ \mu} h_d^{\ \omega} \ .
\end{aligned}$$

With these derivatives, the condition (16) for the gauge Lagrangian is put in the form

$$\begin{aligned}
&\frac{\partial L_0}{\partial R_{\sigma\rho\beta}^{\ \ \ \ \theta}} \left[\delta^\theta_{\ \nu} g_{\gamma\lambda} - \delta^\theta_{\ \gamma} g_{\nu\lambda}\right] R_{\sigma\rho\beta}^{\ \ \ \ \lambda} + \\
&+ \left[\frac{\partial L_0}{\partial \left(\delta_\beta R_{\sigma\rho\alpha}^{\ \ \ \ \gamma}\right)} g_{\nu\lambda} - \frac{\partial L_0}{\partial \left(\delta_\beta R_{\sigma\rho\alpha}^{\ \ \ \ \nu}\right)} g_{\gamma\lambda}\right] \delta_\beta R_{\sigma\rho\alpha}^{\ \ \ \ \lambda} \equiv 0 \ .
\end{aligned} \tag{20}$$

This is a fundamental restriction upon the Lagragians tentatively proposed for the theory, and it is quite useful in order to choose a specific suitable invariant.

4 Quadratic Lagrangian Counting

Our goal here is to determine all possible independent quadratic Lagrangians constructed with the field strength tensors F and G considering their various symmetries. By quadratic Lagrangians we mean invariants of the type FF or GG, but not mixed terms like FG (obviously with the proper contraction of indices). We will also compute the linear case of the Einstein-Hilbert Lagrangian.

4.1 First Order Invariants

The symmetries to be considered in the construction of the invariants of the type FF are those inherited from F: skew-symmetry in each par of indices: $F^{ab}_{\ \ \mu\nu} = -F^{ba}_{\ \ \mu\nu}$ and $F^{ab}_{\ \ \mu\nu} = -F^{ab}_{\ \ \nu\mu}$. Besides these, here is another which is unveiled by the geometrical form of F, eq. (18), namely

$$R_{\sigma\beta\alpha}{}^{\gamma} + R_{\beta\alpha\sigma}{}^{\gamma} + R_{\alpha\sigma\beta}{}^{\gamma} \equiv 0 \ ,$$

the familiar first Bianchi identity met in the context of the general relativity.

Once algebra and space-time indices can be transformed into each other by means of a tetrad, we will consider a compact representation for F:

$$F^{ab}_{\ \ \mu\nu} \to F^{ab}_{\ \ \mu\nu} h_c{}^{\mu} h_d{}^{\nu} \equiv (abcd) \ .$$

Since the Lagrangians are all of the form F^2 with all allowed orders of contractions, it is always possible to rename dummy indices in such a way that the first F will keep its indices in alphabetic order. In the table below it follows all available permutations for the second F:

	Fix. a	Fix. b	Fix. c	Fix. d
cyclic	$(abcd)$	$(bacd)$	$(cabd)$	$(dabc)$
	$(acdb)$	$(bcda)$	$(cbda)$	$(dbca)$
	$(adbc)$	$(bdac)$	$(cdab)$	$(dcab)$
non-cycl.	$(abdc)$	$(badc)$	$(cadb)$	$(dacb)$
	$(acbd)$	$(bcad)$	$(cbad)$	$(dbac)$
	$(adcb)$	$(bdca)$	$(cdba)$	$(dcba)$

(21)

By means of a change in one pair of indices, one can see that the non-cyclic permutations are all proportional to the cyclic ones. Considering only the cyclic permutations and changing two pairs of indices, the table is reduced to:

	Fix. a	Fix. b	Fix. c	Fix. d
cyclical	$(abcd)$	$-$	$-$	$-$
	$(acdb)$	$(bcda)$	$-$	$-$
	$(adbc)$	$(bdac)$	$(cdab)$	$-$

The skew-symmetries of the first F (which has been took in alphabetic order) leads one to restrict once more the possible contractions to the three quadratic invariants

$$\begin{aligned} I_1^F &= (abcd)\,(abcd) \\ I_2^F &= (abcd)\,(acdb) \\ I_3^F &= (abcd)\,(cdab) \quad . \end{aligned} \tag{22}$$

We now analyze invariants constructed with one trace of F. The only non-null type of trace are those obtained by contracting one index of the first pair with one index of the second pair, in view of the skew-symmetry of this object. All possibilities are proportional to

$$TrF \rightarrow h_c{}^{\nu} F^{ca}_{\mu\nu} h_b{}^{\mu} \equiv (\cdot ab \cdot) \quad \text{or} \quad (\circ ab \circ) \quad .$$

The quadratic invariants are given by,

$$\begin{aligned} I_1^{TrF} &= (\cdot ab \cdot)\,(\circ ab \circ) \\ I_2^{TrF} &= (\cdot ab \cdot)\,(\circ ba \circ) \quad . \end{aligned} \tag{23}$$

Still, one can construct a linear invariant taking a double trace of F:

$$I^{TrTrF} = h_c{}^{\nu} F^{ca}_{\;\;\mu\nu} h_a{}^{\mu} \equiv (\cdot \circ \circ \cdot) \ .$$

4.2 Second Order Invariants

Let us introduce a similar notation to the one used in the case of F, i.e.,

$$G^{ab}{}_{\beta\rho\sigma} \rightarrow h_c{}^{\beta} h_d{}^{\rho} h_e{}^{\sigma} G^{ab}{}_{\beta\rho\sigma} \equiv [abcde] \ ,$$

where we identify the following symmetries:

(i) antisymmetry by exchange of indices in the first pair and last two of them,

$$[abcde] = -[bacde] = -[abced] \ ;$$

(ii) Bianchi type identity among the last three indices,

$$[abcde] + [abdec] + [abecd] = 0 \ .$$

4.2.1 Invariants of GG kind

The quadratic combinations are now in a larger amount than in the F^2 case. In fact, we have five tables like (21), one to each letter labeling, since we can associate

$$[abcde] = c\,(abde) \ .$$

Using the symmetries cited above, one finds that the 5! G^2 invariants are reduce to just two kinds:

$$\begin{aligned} I_1^G &= [abcde]\,[abcde] \\ I_2^G &= [abcde]\,[debac] \ . \end{aligned}$$

The detailed and cumbersome calculations are made in the appendix A.

4.2.2 Invariants Involving Traces

There are three independent species of traces for G:

$$\begin{aligned} T^{(1)}_{abc} &= h_d{}^{\beta} G^{da}{}_{\beta\rho\sigma} h_b{}^{\rho} h_c{}^{\sigma} \equiv [\cdot a \cdot bc] \\ T^{(2)}_{abc} &= h_d{}^{\rho} G^{da}{}_{\beta\rho\sigma} h_b{}^{\beta} h_c{}^{\sigma} \equiv [\cdot ab \cdot c] \\ T^{(3)}_{abc} &= g^{\beta\rho} G^{ab}{}_{\beta\rho\sigma} h_c{}^{\sigma} \equiv [ab \cdot \cdot c] \ . \end{aligned} \tag{24}$$

Again using symmetries (see appendix A) we arrive at:

$$
\begin{array}{|c|c|}
\hline
Tr\mathcal{G}_3 = [\cdot ab \cdot c]\,[\cdot a \cdot bc] & Tr\mathcal{G}_{11} = [\cdot ab \cdot c]\,[\cdot bc \cdot a] \\
\hline
Tr\mathcal{G}_5 = [\cdot ab \cdot c]\,[\cdot c \cdot ab] & Tr\mathcal{G}_{14} = [\cdot ab \cdot c]\,[\cdot ab \cdot c] \\
\hline
Tr\mathcal{G}_6 = [ab \cdot \cdot c]\,[\cdot a \cdot bc] & Tr\mathcal{G}_{17} = [ab \cdot \cdot c]\,[ab \cdot \cdot c] \\
\hline
Tr\mathcal{G}_{10} = [\cdot ab \cdot c]\,[\cdot ba \cdot c] & Tr\mathcal{G}_{18} = [ab \cdot \cdot c]\,[ac \cdot \cdot b] \\
\hline
\end{array}
\tag{25}
$$

while for double traces we have:

$$
\begin{aligned}
TrTr\mathcal{G}_1 &= [\cdot \circ b \cdot \circ]\,[\cdot \circ b \cdot \circ] \\
TrTr\mathcal{G}_2 &= [\circ b \cdot \cdot \circ]\,[\circ b \cdot \cdot \circ] \\
TrTr\mathcal{G}_3 &= [\cdot \circ b \cdot \circ]\,[\circ b \cdot \cdot \circ] \ .
\end{aligned}
\tag{26}
$$

4.3 Bianchi Identities

As we already said, until now we have not used the first Bianchi identity:

$$R^{\sigma\chi\rho\beta} + R^{\chi\rho\sigma\beta} + R^{\rho\sigma\chi\beta} \equiv 0 \ . \tag{27}$$

In geometrical variables, the cyclic property of G is translated to the second Bianchi identity:

$$\delta^{\mu} R^{\sigma\chi\rho\beta} + \delta^{\sigma} R^{\chi\mu\rho\beta} + \delta^{\chi} R^{\mu\sigma\rho\beta} \equiv 0 \ .$$

These identities can reduce the number of independent invariants, since $F \propto R$ and $G \propto \delta R$.

4.3.1 Reducing invariants

Let us begin by invariants of form F^2. The first three are (22):

$$
\begin{aligned}
I_1^F &= R_{\sigma\rho\chi\kappa} R^{\sigma\rho\chi\kappa} \\
I_2^F &= R_{\sigma\rho\chi\kappa} R^{\sigma\chi\rho\kappa} \\
I_3^F &= R_{\sigma\rho\chi\kappa} R^{\chi\kappa\sigma\rho} \ .
\end{aligned}
$$

As a consequence of the first Bianchi identity (27) and the skewsymmetries, the curvature tensor obey:

$$R_{\sigma\rho\chi\kappa} = R_{\chi\kappa\sigma\rho} \ . \tag{28}$$

Then,

$$I_3^F = R_{\sigma\rho\chi\kappa}R^{\chi\kappa\sigma\rho} = R_{\sigma\rho\chi\kappa}R^{\sigma\rho\chi\kappa} = I_1^F \ ,$$

while for I_2^F one finds,

$$\begin{aligned} I_2^F &= R_{\sigma\rho\chi\kappa}R^{\sigma\chi\rho\kappa} = -\left(R_{\rho\chi\sigma\kappa} + R_{\chi\sigma\rho\kappa}\right) R^{\sigma\chi\rho\kappa} = \\ &= -R_{\chi\rho\sigma\kappa}R^{\chi\sigma\rho\kappa} + R_{\chi\sigma\rho\kappa}R^{\chi\sigma\rho\kappa} = \\ &= -I_2^F + I_1^F \ , \end{aligned}$$

$$2I_2^F = I_1^F \ ,$$

which let us with only one invariant of this kind, I_1^F.

Now, we translate the trace-like invariants (23) in a geometrical form:

$$\begin{aligned} I_1^{TrF} &= R_{\rho\mu\nu}{}^{\rho} R_{\sigma}{}^{\mu\nu\sigma} \\ I_2^{TrF} &= R_{\rho\mu\nu}{}^{\rho} R_{\sigma}{}^{\nu\mu\sigma} \ . \end{aligned}$$

Since the Ricci tensor $R_{\mu\nu} \equiv R_{\rho\mu\nu}{}^{\rho}$ is symmetric,[1] we have in fact only one invariant, $I_1^{TrF} = R_{\mu\nu}R^{\mu\nu}$.

At least, the only invariant of double traced form in F is:

$$I^{TrTrF} = R \ .$$

Analogously, in view of the Bianchi identities, only four invariants of the type G^2 remains (see appendix A):

$I_1^G = \delta_\beta R_{\sigma\rho\chi\kappa}\delta^\beta R^{\sigma\rho\chi\kappa}$	$TrTr\mathcal{G}_2 = \delta^\rho R_{\rho\chi}\delta_\mu R^{\mu\chi}$
$Tr\mathcal{G}_{10} = \delta_\beta R_{\sigma\chi}\delta^\chi R^{\sigma\beta}$	$Tr\mathcal{G}_{14} = \delta_\beta R_{\sigma\chi}\delta^\beta R^{\sigma\chi}$

5 Gauge Invariance Condition

With the invariants constructed above we collect seven types of Lagrangians for the gravitational field:

[1] Which is a consequence of the first Bianchi identity.

Lagrangians	Invariants	Gauge Form	Geom. Form
$L_0^{(R_1)}$	$h\left(I^{TrTrF}\right)^n$	$h\left(h_a{}^\nu h_b{}^\mu F^{ab}{}_{\mu\nu}\right)^n$	$hR^n\ ,\ n=1,2.$
$L_0^{(R_2)}$	$h\left(I_1^{TrF}\right)$	$hh_a{}^\nu h^c{}_\sigma F^a{}_{b\mu\nu}F_c{}^{b\mu\sigma}$	$hR_{\mu\nu}R^{\mu\nu}$
$L_0^{(R_3)}$	$h\left(I_1^F\right)$	$hF^{ab}{}_{\mu\nu}F_{ab}{}^{\mu\nu}$	$hR_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma}$
$L_0^{(G_1)}$	$h\left(TrTr\mathcal{G}_2\right)$	$hh^a{}_\sigma h_c{}^\nu G_{ab\beta}{}^{\beta\sigma}G^{cb\mu}{}_{\mu\nu}$	$h\delta^\rho R_{\rho\chi}\delta_\mu R^{\mu\chi}$
$L_0^{(G_2)}$	$h\left(Tr\mathcal{G}_{14}\right)$	$hh_a{}^\rho h^c{}_\nu G^{ab}{}_{\mu\rho\sigma}G_{cb}{}^{\mu\nu\sigma}$	$h\delta_\beta R_{\sigma\chi}\delta^\beta R^{\sigma\chi}$
$L_0^{(G_3)}$	$h\left(Tr\mathcal{G}_{10}\right)$	$hh_a{}^\rho h^d{}_\nu h_b{}^\mu h^e{}_\beta G^{ab\beta}{}_{\rho\sigma}G_{de\mu}{}^{\nu\sigma}$	$h\delta_\beta R_{\sigma\chi}\delta^\chi R^{\sigma\beta}$
$L_0^{(G_4)}$	$h\left(I_1^G\right)$	$hG^{ab}{}_{\mu\nu\lambda}G_{ab}{}^{\mu\nu\lambda}$	$h\delta_\beta R_{\sigma\rho\chi\kappa}\delta^\beta R^{\sigma\rho\chi\kappa}$

(29)

We are considering Lagrangians only up to quadratic order in F and or G, which also include the linear invariant $I^{TrTrF}=R$ and their square R^2. Actually, one can observe that if any invariant fulfills the gauge invariance condition, then any of its power will also do it, since this condition is linear in the derivatives $\frac{\partial L_0}{\partial F}$ and $\frac{\partial L_0}{\partial G}$. For instance,

$$L_0 = I^n\ ,\ \frac{\partial L_0}{\partial F} = nI^{n-1}\frac{\partial I}{\partial F}\ .$$

Therefore,

$$\frac{\partial I}{\partial F}\left[...\right]F=0 \Rightarrow \frac{\partial L_0}{\partial F}\left[...\right]F=0\ ,$$

and the same follows for G.

Using the skewsymmetry $\nu \leftrightarrow \gamma$ of the equation (20) and the symmetry properties of the Riemann tensor, one can easily verify that all Lagrangians densities listed in (29) accomplish the gauge invariance condition. Then, any function of these invariants expressible in a Taylor series also will fulfill the gauge invariance condition.

6 Equations of Motion

Here we will concentrate our attention on the effect of the term

$$L_0^{(G_1)} = \frac{1}{2}hh^a{}_\sigma h_c{}^\nu G_{ab\beta}{}^{\beta\sigma}G^{cb\mu}{}_{\mu\nu} = \frac{1}{8}h\delta^\rho R\delta_\rho R$$

on a gravitational theory based on the Einstein-Hilbert action plus the $L_0^{(G_1)}$ term. This Lagrangian density is equivalent, by the Bianchi identity, to the form $\frac{1}{2}h\delta^\rho R_{\rho\chi}\delta_\mu R^{\mu\chi}$, which is clearly analogous to Podolsky's second order term for Electrodynamics ($L_{Podolsky} \propto \partial^\rho F_{\rho\chi}\partial_\mu F^{\mu\chi}$). The choice of the particular Lagrangian $L_0^{(G_1)}$ is only motivated by this analogy. Besides this, the $L_0^{(G_1)}$ term also can be viewed as a kind of kinetic term for the scalar curvature, what approximate such description to the usual scalar fields.

Taking a functional variation of the tetrad field, one finds:

$$h = \sqrt{-g}\ , \quad \delta h = \frac{1}{2}hg^{\lambda\nu}\delta g_{\lambda\nu} = hg^{\lambda\nu}h^a{}_\lambda\eta_{ab}\delta h^b{}_\nu$$

and

$$\begin{aligned}\delta L_0^{(G_1)} = &\frac{1}{4}\partial_\rho\left(h\partial^\rho R\delta R\right) - \frac{1}{4}\delta R\partial_\rho\left(h\partial^\rho R\right) + \\ &+\frac{1}{4}h\left[\frac{1}{2}g^{\lambda\nu}\partial^\rho R\partial_\rho R - g^{\mu\nu}g^{\rho\lambda}\partial_\mu R\partial_\rho R\right]h^a{}_\lambda\eta_{ab}\delta h^b{}_\nu\ .\end{aligned}$$

On calculating the equations of motion, we must give special attention to the last term involving

$$\delta R = -2R_{\mu\beta}g^{\mu\nu}g^{\beta\lambda}h^a{}_\lambda\eta_{ab}\delta h^b{}_\nu + \frac{1}{h}\partial_\alpha\left[h\left(g^{\mu\nu}\delta\Gamma^\alpha_{\mu\nu} - g^{\nu\alpha}\delta\Gamma^\beta_{\nu\beta}\right)\right]\ ,$$

which will include several integration by parts. After these integrations and some cumbersome calculations, one finds:

$$\begin{aligned}\delta L_0^{(G_1)} = &\frac{1}{2}\partial_\theta\mathcal{V}^\theta + \frac{1}{2}h\left[\delta_\lambda\delta_\nu\left[\Diamond R\right] + \frac{1}{2}\delta_\lambda R\delta_\nu R - R_{\lambda\nu}\Diamond R + \right. \\ &\left. - g_{\lambda\nu}\Diamond\left[\Diamond R\right] - \frac{1}{4}g_{\lambda\nu}\delta^\rho R\delta_\rho R\right]h_a{}^\lambda\eta^{ab}\delta h_b{}^\nu\ ,\end{aligned}$$

where

$$\begin{aligned}\mathcal{V}^\theta \equiv &-\frac{1}{2}\left(g^{\mu\nu}\delta\Gamma^\theta_{\mu\nu} - g^{\nu\theta}\delta\Gamma^\beta_{\nu\beta}\right)\partial_\rho\left(h\partial^\rho R\right) - \frac{1}{2}h\partial^\theta R\delta R + \\ &+\frac{1}{4}h\left(g^{\mu\nu}\delta^\alpha_\sigma - g^{\nu\alpha}\delta^\mu_\sigma\right)\delta_\alpha\left[\Diamond R\right]g^{\sigma\beta}\left(\delta^\theta{}_\nu\delta^\lambda{}_\mu\delta^\eta{}_\beta + \delta^\theta{}_\mu\delta^\lambda{}_\beta\delta^\eta{}_\nu - \delta^\theta{}_\beta\delta^\lambda{}_\nu\delta^\eta{}_\mu\right)\delta g_{\lambda\eta}\ ,\end{aligned}$$

and

$$\Diamond \equiv \delta_\beta\delta^\beta$$

is the Laplace-Beltrami operator on the Riemannian space.

Therefore, the second order contribution to the equation of motion will be

$$H^b_\nu \equiv h^{b\lambda}\delta_\lambda\delta_\nu\left[\Diamond R\right] + \frac{1}{2}h^{b\lambda}\delta_\lambda R\delta_\nu R - R^b{}_\nu\Diamond R - h^b{}_\nu\delta_\beta\delta^\beta\left[\Diamond R\right] - \frac{1}{4}h^b{}_\nu\delta^\rho R\delta_\rho R\ . \tag{30}$$

Furthermore, if we include the usual first order Einstein-Hilbert and a matter Lagrangian densities,

$$S_T = \int d^n x \left(-\frac{hR}{2\chi} - \frac{\beta}{\chi} L_0^{(G_1)} + h\mathcal{L}_{matter} \right) \ ,$$

the field equations become

$$G^b{}_\nu + \beta H^b_\nu = \chi T^b{}_\nu \ , \tag{31}$$

or in a geometrical form,

$$\begin{aligned} R_{\lambda\nu} - \frac{1}{2} g_{\lambda\nu} R + \beta \Big[\delta_\lambda \delta_\nu \left(\Diamond R \right) + \frac{1}{2} \delta_\lambda R \delta_\nu R + \\ - R_{\lambda\nu} \Diamond R - g_{\lambda\nu} \Diamond \left(\Diamond R \right) - \frac{1}{4} g_{\lambda\nu} \delta^\rho R \delta_\rho R \Big] = \chi T_{\lambda\nu} \ , \end{aligned}$$

where $G^b{}_\nu$ is the Einstein tensor and

$$T_{\lambda\nu} \equiv \frac{2}{h} \frac{\delta \left(h\mathcal{L}_{matter} \right)}{\delta g^{\lambda\nu}}$$

is the energy-momentum tensor of the matter fields written in terms of the metric field.

By analogy to the Alekseev-Arbuzhov-Baikov [11], one could expect that the higher order terms, which can be until sixth derivative order, would be related to infrared corrections to General Relativity, giving sensible physical effects at large scales.

6.1 Covariant Conservation of $T_{\lambda\nu}$

Taking the covariant divergence of (31), we have

$$\delta^\nu G_{\nu\alpha} + \beta \delta^\nu H_{\nu\alpha} = \chi \delta^\nu T_{\nu\alpha} \ .$$

Now, from the first order case, we know that

$$\delta^\nu G_{\nu\alpha} \equiv 0 \ .$$

Applying the divergence to the equation (30), one finds

$$\begin{aligned} \delta^\nu H_{\nu\alpha} = \delta^\nu \delta_\nu \delta_\alpha \Diamond R - g_{\nu\alpha} \delta^\nu \Diamond \left[\Diamond R \right] + \frac{1}{2} \delta_\nu R \delta^\nu \delta_\alpha R + \frac{1}{2} \delta^\nu \delta_\nu R \delta_\alpha R + \\ - \delta^\nu R_{\nu\alpha} \Diamond R - R_{\nu\alpha} \delta^\nu \Diamond R - \frac{1}{4} g_{\nu\alpha} \delta^\nu \left(\delta^\rho R \delta_\rho R \right) \ . \end{aligned}$$

Using the commutation relation

$$[\delta_\nu, \delta_\alpha] A^\tau = R_{\alpha\nu}{}^{\tau\xi} A_\xi$$

and the second Bianchi identity, we arrive at

$$\delta^\nu H_{\nu\alpha} = R_{\alpha\xi}\delta^\xi \Diamond R - R_{\nu\alpha}\delta^\nu \Diamond R + \frac{1}{2}\delta_\nu R \delta^\nu \delta_\alpha R - \frac{1}{2}\delta^\rho R \delta_\alpha \delta_\rho R = 0 \ .$$

Then, the covariant conservation of $T_{\mu\nu}$ is established:

$$\delta^\mu \left(G_{\mu\nu} + \beta H_{\mu\nu}\right) \equiv 0 \Longrightarrow \delta^\mu T_{\mu\nu} = 0 \ .$$

7 Conclusion

We have applied the second order gauge theory [10] to the local gauge theory for the homogeneous Poincaré group. It was found that the geometrical counterparts of the usual field strength F and the second order field strength $G = DF$ are the Riemann tensor R and its (space-time) covariant derivative δR. It followed the analysis of the second order invariants composed with the geometrical entities.

We demonstrate – employing the symmetry properties of the curvature tensor – that the only independent Lagrangian densities for the gravitational field in a Riemannian manifold of arbitrary dimension are the seven ones listed in table (29). Linear combinations of terms proportional to powers of R, as the familiar quadratic term in the curvature, are of first order in the gauge potential ω, therefore, in the context of the second order gauge theory, the contributions of second order in the Lagrangian density, which are those including second derivatives of the gauge potential, are of type δR.

Using a particularly simple choice for the second order gauge Lagrangian – inspired in the Podolsky's proposal for a Generalized Electrodynamics, we derived equations of motion which are naturally compatible with the covariant conservation of the energy-momentum tensor.

In the future, we will study the static and isotropic solution of these field equations, searching for massive modes which do not violate the local gauge symmetry. Our guide in these calculations will by the treatment given in [10] to the $U(1)$ case, where an effective mass for the photon was derived. To do this, one naturally must concern about the determination of the conserved current associated with the local Lorentz symmetry and the relationship to the global diffeomorphic invariance of the theory.

Another perspective is to apply the second order equations of motion (31) to a Friedmann-Robertson-Walker metric. The goal is to seek for a accelerated regimes of the cosmological model arising from the higher order terms. This proposal is now under investigation.

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A Appendix: Counting Second Order Invariants

A.1 Counting GG Invariants

First, let us analyse how many are the possible contractions of kind GG. This is done by means of tables as in the section 4.1. The first one is constructed fixing, for instance, the last index:

Fix. e	Fix. a	Fix. b	Fix. c	Fix. d
cyclic	$(abcd)$	$(bacd)$	$(cabd)$	$(dabc)$
	$(acdb)$	$(bcda)$	$(cbda)$	$(dbca)$
	$(adbc)$	$(bdac)$	$(cdab)$	$(dcab)$
non-cycl.	$(abdc)$	$(badc)$	$(cadb)$	$(dacb)$
	$(acbd)$	$(bcad)$	$(cbad)$	$(dbac)$
	$(adcb)$	$(bdca)$	$(cdba)$	$(dcba)$

Analogous tables result when we fix the indices d,c,b and a. For each table, non-cyclic permutations are equivalent to cyclic ones, giving:

Fix. e	Fix. a	Fix. b	Fix. c	Fix. d
cyclic	$(abcd)$	–	–	–
	$(acdb)$	$(bcda)$	–	–
	$(adbc)$	$(bdac)$	$(cdab)$	–

an similarly for the other four tables.

Using the cyclic permutation symmetry, one can identify elements of different tables, reducing the number of invariants. By the skew-symmetry in the first G and renaming dummy indices, it follows:

$\mathcal{G}_1 = [abcde]\,[abcde]$	$\mathcal{G}_6 = [abcde]\,[cdbea]$
$\mathcal{G}_2 = [abcde]\,[beacd]$	$\mathcal{G}_7 = [abcde]\,[adbec]$
$\mathcal{G}_3 = [abcde]\,[adceb]$	$\mathcal{G}_8 = [abcde]\,[acbde]$
$\mathcal{G}_4 = [abcde]\,[aecbd]$	$\mathcal{G}_9 = [abcde]\,[acdeb]$
$\mathcal{G}_5 = [abcde]\,[debac]$	$\mathcal{G}_{10} = [abcde]\,[abdce]$

One can further apply the cyclic permutation symmetry to the first G in these remaining invariants and reduce even more the number of independent quantities. Beginning with $\mathcal{G}_{10}$:

$$\mathcal{G}_{10} = -\left([abdec] + [abecd]\right)[abdce] = \mathcal{G}_1 - \mathcal{G}_{10} \Rightarrow 2\mathcal{G}_{10} = \mathcal{G}_1 \ .$$

On the other hand, for $\mathcal{G}_9$:

$$\mathcal{G}_9 = -\left([abdec] + [abecd]\right)[acdeb] = 2\mathcal{G}_4 + \mathcal{G}_9 \ \Rightarrow \mathcal{G}_4 = 0 \ .$$

Proceeding in the same way, one finds the following identities:

$$\begin{aligned} &2\mathcal{G}_{10} = \mathcal{G}_1; \qquad\qquad 2\mathcal{G}_6 = \mathcal{G}_5 \ ; \\ &\mathcal{G}_2 = \mathcal{G}_3 = \mathcal{G}_4 = \mathcal{G}_7 = \mathcal{G}_8 = \mathcal{G}_9 = 0 \ . \end{aligned}$$

which give two independent invariants,

$$I_1^G = [abcde]\,[abcde] \ , \qquad I_2^G = [abcde]\,[debac] \ .$$

A.2 Counting $(TrG)^2$ Invariants

Starting with the three independet traces listed in (24), and considering skew-symmetries, the possible quadratic combinations are:

$Tr\mathcal{G}_1 = [\cdot a \cdot bc]\,[\cdot a \cdot bc]$	$Tr\mathcal{G}_7 = [ab \cdot \cdot c]\,[\cdot c \cdot ab]$	$Tr\mathcal{G}_{13} = [\cdot ab \cdot c]\,[\cdot cb \cdot a]$
$Tr\mathcal{G}_2 = [\cdot a \cdot bc]\,[\cdot b \cdot ac]$	$Tr\mathcal{G}_8 = [\cdot ab \cdot c]\,[\cdot ab \cdot c]$	$Tr\mathcal{G}_{14} = [\cdot ab \cdot c]\,[ab \cdot \cdot c]$
$Tr\mathcal{G}_3 = [\cdot ab \cdot c]\,[\cdot a \cdot bc]$	$Tr\mathcal{G}_9 = [\cdot ab \cdot c]\,[\cdot ac \cdot b]$	$Tr\mathcal{G}_{15} = [\cdot ab \cdot c]\,[ac \cdot \cdot b]$
$Tr\mathcal{G}_4 = [\cdot ab \cdot c]\,[\cdot b \cdot ac]$	$Tr\mathcal{G}_{10} = [\cdot ab \cdot c]\,[\cdot ba \cdot c]$	$Tr\mathcal{G}_{16} = [\cdot ab \cdot c]\,[bc \cdot \cdot a]$
$Tr\mathcal{G}_5 = [\cdot ab \cdot c]\,[\cdot c \cdot ab]$	$Tr\mathcal{G}_{11} = [\cdot ab \cdot c]\,[\cdot bc \cdot a]$	$Tr\mathcal{G}_{17} = [ab \cdot \cdot c]\,[ab \cdot \cdot c]$
$Tr\mathcal{G}_6 = [ab \cdot \cdot c]\,[\cdot a \cdot bc]$	$Tr\mathcal{G}_{12} = [\cdot ab \cdot c]\,[\cdot ca \cdot b]$	$Tr\mathcal{G}_{18} = [ab \cdot \cdot c]\,[ac \cdot \cdot b]$

The last two invariants can not be converted into any other using the symmetries at our disposal. Each one of the preceding $Tr\mathcal{G}$ must be analyzed, case by case, in a search for eventual interdependence.

Take, for example, the 16^{th} term, and rewrite it as bellow:

$$Tr\mathcal{G}_{16} = -\,[\cdot acb\cdot]\,[bc \cdot \cdot a] - [\cdot a \cdot cb]\,[bc \cdot \cdot a] \Rightarrow 2Tr\mathcal{G}_{16} = Tr\mathcal{G}_7 \ .$$

Repeat the reasoning for, say, the 15^{th} invariant:

$$Tr\mathcal{G}_{15} = -\,[\cdot acb\cdot]\,[ac \cdot \cdot b] - [\cdot a \cdot cb]\,[ac \cdot \cdot b] = Tr\mathcal{G}_{14} - Tr\mathcal{G}_6 \ .$$

As soon as we perform this same check for all the above invariants, only eight of them are kept:

$Tr\mathcal{G}_3 = [\cdot ab \cdot c]\,[\cdot a \cdot bc]$	$Tr\mathcal{G}_{11} = [\cdot ab \cdot c]\,[\cdot bc \cdot a]$
$Tr\mathcal{G}_5 = [\cdot ab \cdot c]\,[\cdot c \cdot ab]$	$Tr\mathcal{G}_{14} = [\cdot ab \cdot c]\,[\cdot ab \cdot c]$
$Tr\mathcal{G}_6 = [ab \cdot \cdot c]\,[\cdot a \cdot bc]$	$Tr\mathcal{G}_{17} = [ab \cdot \cdot c]\,[ab \cdot \cdot c]$
$Tr\mathcal{G}_{10} = [\cdot ab \cdot c]\,[\cdot ba \cdot c]$	$Tr\mathcal{G}_{18} = [ab \cdot \cdot c]\,[ac \cdot \cdot b]$

. (A.1)

A.3 Counting $(TrTrG)^2$ Invariants

From $T^{(1)}_{abc} \equiv [\cdot a \cdot bc]$ one can take a trace again:

$$T^{(1)}_c \equiv [\cdot \circ \cdot \circ c] \ .$$

From $T^{(2)}_{abc} \equiv [\cdot ab \cdot c]$ one finds $T_c \equiv [\cdot \circ \circ \cdot c]$ which can be reduced to $T^{(1)}_c$ using the G skewsymmetry in the first two indexes and changing dummy indexes. Another possible trace is constructed from $T^{(2)}_{abc}$:

$$T^{(2)}_b \equiv [\cdot \circ b \cdot \circ] \ . \tag{A.2}$$

But it also is not independent of $T^{(1)}_c$:

$$T^{(2)}_b \equiv [\cdot \circ b \cdot \circ] = -[\cdot \circ \cdot \circ b] - [\cdot \circ \circ b \cdot] = -T^{(1)}_b - [\circ \cdot \circ \cdot b] = -2T^{(1)}_b \ .$$

Let us set $T^{(2)}_b$ as the independent double trace.

There are an internal double trace of $T^{(3)}_{abc} \equiv [ab \cdot \cdot c]$ which is independent of $T^{(2)}_c$:

$$T^{(3)}_b \equiv [\circ b \cdot \cdot \circ] \ . \tag{A.3}$$

The other double trace of $T^{(3)}_{abc}$ is,

$$T_b \equiv [b \circ \cdot \cdot \circ] = -T^{(3)}_b \ .$$

Then, we have the following set of independent double traces:

$$\begin{aligned} TrTr\mathcal{G}_1 &= [\cdot \circ b \cdot \circ]\,[\cdot \circ b \cdot \circ] \\ TrTr\mathcal{G}_2 &= [\circ b \cdot \cdot \circ]\,[\circ b \cdot \cdot \circ] \\ TrTr\mathcal{G}_3 &= [\cdot \circ b \cdot \circ]\,[\circ b \cdot \cdot \circ] \ . \end{aligned} \tag{A.4}$$

A.4 Reducing the G^2 invariants using Bianchi identities

Consider the reduction of the number of quadratic invariants in G by means of Bianchi identities. Using the geometric form, the first two invariants are:

$$I^G_1 = \delta_\beta R_{\sigma\rho\chi\kappa} \delta^\beta R^{\sigma\rho\chi\kappa} \ , \qquad I^G_2 = \delta_\beta R_{\sigma\rho\chi\kappa} \delta^\chi R^{\beta\kappa\sigma\rho} \ .$$

Applying the second Bianchi identity to I^G_2 we have:

$$I^G_2 = -\delta_\beta R_{\sigma\rho\chi\kappa} \left(\delta^\chi R^{\kappa\beta\rho\sigma} + \delta^\beta R^{\chi\kappa\rho\sigma} \right) = I^G_1 - I^G_2 \Rightarrow 2I^G_2 = I^G_1 \ ,$$

therefore it is sufficient to consider only I^G_1.

Let us analyse now the trace invariants in G, (25):

$Tr\mathcal{G}_3 = \delta_\beta R_{\sigma\zeta\chi}{}^{\zeta}\delta_\mu R^{\sigma\beta\chi\mu}$	$Tr\mathcal{G}_{11} = \delta_\beta R_{\sigma\chi}\delta^\sigma R^{\chi\beta}$
$Tr\mathcal{G}_5 = \delta_\beta R_{\sigma\rho\chi}{}^{\rho}\delta_\mu R^{\beta\chi\sigma\mu}$	$Tr\mathcal{G}_{14} = \delta_\beta R_{\sigma\chi}\delta^\beta R^{\sigma\chi}$
$Tr\mathcal{G}_6 = \delta^\rho R_{\sigma\rho\chi\zeta}\delta_\kappa R^{\sigma\chi\zeta\kappa}$	$Tr\mathcal{G}_{17} = \delta^\rho R_{\sigma\rho\chi\kappa}\delta_\mu R^{\sigma\mu\chi\kappa}$
$Tr\mathcal{G}_{10} = \delta_\beta R_{\sigma\chi}\delta^\chi R^{\sigma\beta}$	$Tr\mathcal{G}_{18} = \delta^\rho R_{\sigma\rho\chi\kappa}\delta_\mu R^{\chi\mu\sigma\kappa}$

Comparing $Tr\mathcal{G}_{10}$ with $Tr\mathcal{G}_{11}$ one sees that both are the same invariant, due to the symmetry of Ricci tensor.

Using the second Bianchi identity, it follows:

$$Tr\mathcal{G}_3 = \delta_\beta R_{\sigma\chi} g_{\mu\rho}\left(\delta^\beta R^{\rho\sigma\chi\mu} + \delta^\sigma R^{\beta\rho\chi\mu}\right) = Tr\mathcal{G}_{14} - Tr\mathcal{G}_{10}$$

and in the same way,

$$Tr\mathcal{G}_5 = Tr\mathcal{G}_6 = -\frac{1}{2}Tr\mathcal{G}_{17} = -Tr\mathcal{G}_{18} = -Tr\mathcal{G}_3 \ .$$

This shows that only $Tr\mathcal{G}_{14}$ and $Tr\mathcal{G}_{10}$ can be hold independent.

We apply the same technique to the double traced invariants (26):

$$TrTr\mathcal{G}_1 = \delta_\beta R\delta^\beta R \ , \quad TrTr\mathcal{G}_2 = \delta^\rho R_{\rho\chi}\delta_\mu R^{\mu\chi} \ , \quad TrTr\mathcal{G}_3 = -\delta_\beta R\delta_\mu R^{\mu\beta} \ .$$

The second Bianchi identity shows us that we have only one invariant in such case:

$$TrTr\mathcal{G}_3 = \delta^\rho{}_\zeta g^{\sigma\chi}\left(\delta_\sigma R_{\beta\rho\chi}{}^{\zeta} + \delta_\rho R_{\sigma\beta\chi}{}^{\zeta}\right)\delta_\mu R^{\mu\beta} = -2TrTr\mathcal{G}_2 = -\frac{1}{2}TrTr\mathcal{G}_1 \ .$$

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